Quantum tunnelling of a damped and driven, inverted harmonic oscillator

1993 J. Phys. A: Math. Gen. 26 7137

(http://iopscience.iop.org/0305-4470/26/23/048)

J. Phys. A: Math. Gen. 26 (1993) 7137–7147. Printed in the UK

Quantum tunnelling of a damped and driven, inverted harmonic oscillator

S Baskoutas†‡||, A Jannussis†‡|| and R Mignani§‡

† Dipartimento di Fisica, Ia Universitá di Roma 'La Sapienza' P le A Moro, 2-I-00185 Rome, Italy

‡ INFN-Sezione di Roma, c/o Dipartimento di Fisica Ia Universitá di Roma 'La Sapienza', P le A Moro, 2-I-00185 Rome, Italy

§ Dipartimento di Fisica 'E Amaldi', IIIa Universitá di Roma, Rome, Italy

Received 27 January 1993, in final form 26 April 1993

Abstract. Using the evolution operator method, we derive the exact propagator of the generalized parametric oscillator in its more general form. This result is exploited to obtain the exact wavefunction of a damped and driven, inverted harmonic oscillator of the Caldirola–Kanai type, taking a Gaussian wavepacket as the initial state. We discuss the tunnelling process of such a system. The probability density and the persistence probability are evaluated. The expression for the sojourn time is derived for a small external force, and is the sum of two terms, whose explicit forms are obtained in the case of an extended wavepacket. The first term is an increasing function of the dissipation parameter γ, whereas the second one is strictly due to the presence of the driving force.

1. Introduction

In the last few decades there has been sustained interest in quantum systems with time-dependent Hamiltonians. Indeed, as is well known, they provide a phenomenological description of dissipative processes, and are widely used to model phenomena in which interactions with the surroundings play a basic role. In particular, some methods have been derived in order to obtain exact solutions of the Schrödinger equation for time-dependent oscillators. The main tools are: (i) path-integral formalism [1]; (ii) dynamical invariants [2, 3]; (iii) time-dependent canonical transformations [3]; (iv) second quantization [4, 5]; and (v) group-theoretical methods [6–8].

The last approach is based on the determination of the evolution operator by means of the time-ordering algebraic method developed by Wei and Norman [9]. It can be applied whenever the Hamiltonian can be expressed as a (time-dependent) linear combination of the generators of some Lie group.

The Lie algebraic method has been applied, among others, to the study of a damped harmonic oscillator driven by a time-dependent external force, by exploiting the underlying $SU(1,1) \oplus h(4)$ structure of its Hamiltonian [10, 11]. The same problem has been dealt with in [5] by other methods. We recall that Hamiltonians with such an algebraic structure have been considered in a variety of physical problems, e.g. pulse propagation in a free-electron laser [10], the analysis of time-dependent coherent states [11] and the generation of non-Poissonian effects in laser-plasma scattering [12].

|| On leave from Department of Physics, Patras University, 26110 Patras, Greece (permanent address) and IBR, Palm Harbor, FL 34682, USA.

In the present paper we study the problem of quantum tunnelling for a damped, inverted harmonic oscillator driven by a monochromatic external force, with a Hamiltonian of the form

$$\hat{H}(t) = \frac{\hat{p}^2}{2m}\mathrm{e}^{-\gamma t} - \frac{m}{2}\omega^2\hat{q}^2\mathrm{e}^{\gamma t} + \lambda\hat{q}\cos\bar{\Omega}t \tag{1.1}$$

whose exact wavefunctions are derived by just exploiting its $SU(1,1) \oplus h(4)$ algebraic structure. As is well known, the first part of Hamiltonian (1.1) without the external term is called the inverted Caldirola–Kanai (CK) Hamiltonian, and is formally obtained from the usual CK Hamiltonian [13] by the replacement $\omega \to \mathrm{i}\omega$. In spite of the many analogies between the two Hamiltonians, the physics is different: the energy eigenstates of the inverted one are not square integrable and are doubly degenerate, e.g. with respect to incidence from the left or right or alternatively with respect to parity [14].

Of course, for $\gamma \to 0$ the inverted CK Hamiltonian reduces to that of the inverted harmonic oscillator, which has been applied, for example, to masers [15] and to reactive scattering [16].

Both the inverted oscillator and inverted CK Hamiltonians produce squeezed states [14, 17]. We also recall that the problem of quantum tunnelling for the inverted CK Hamiltonian has recently been discussed by two of us [18].

The plan of this paper is as follows. In section 2 we derive the expression for the propagator of the generalized quadratic parametric oscillator with the Hamiltonian

$$\hat{H}(t) = \tfrac{1}{2}\left[Z(t)\frac{\hat{p}^2}{m} + \omega Y(t)(\hat{p}\hat{q} + \hat{q}\hat{p}) + X(t)m\omega^2\hat{q}^2\right] + \mu(t)\hat{q} + \nu(t)\hat{p} \tag{1.2}$$

by exploiting its $SU(1,1) \oplus h(4)$ structure and the Wei–Norman (WN) method. The results of section 2 are applied in section 3 to finding the exact wavefunctions of system (1.1) for an initial Gaussian wavepacket. In section 4 we calculate the probability density, the persistence probability and, then, the sojourn (or dwell) time, which (for a small perturbation) is the sum of two terms. Section 5 concludes the paper. The explicit calculation of the sojourn time for an extended wavepacket is given in the appendix.

2. Propagator of the generalized quadratic parametric oscillator

Consider the generalized quadratic parametric oscillator [19]

$$\hat{H}(t) = \hat{H}_0(t) + \hat{V}(t) \tag{2.1}$$

where

$$\hat{H}(t) = \tfrac{1}{2}\left[Z(t)\frac{\hat{p}^2}{m} + \omega Y(t)(\hat{p}\hat{q} + \hat{q}\hat{p}) + X(t)m\omega^2\hat{q}^2\right] \tag{2.2}$$

and

$$\hat{V}(t) = \mu(t)\hat{q} + \nu(t)\hat{p}. \tag{2.3}$$

The problem of finding the evolution operator, in the form of a WN ordered product, for a Hamiltonian with an $SU(1,1) \oplus h(4)$ algebraic structure has been solved formally in [7] and [10] by applying the Levi theorem. In the present case, we can take advantage of knowing the 'unperturbed' evolution operator (corresponding to the Hamiltonian $\hat{H}_0(t)$) [20], the explicit coordinate representation of the generators of the Weyl group and the one-dimensional Lorentz group. Indeed, we can express the evolution operator corresponding to $\hat{H}(t)$ as

$$\hat{U}(t) = \hat{U}_0(t)\hat{U}_I(t) \tag{2.4}$$

where $\hat{U}_0(t)$ and $\hat{U}_I(t)$ satisfy the equations

$$\mathrm{i}\hbar\frac{\partial \hat{U}_0(t)}{\partial t} = \hat{H}_0(t)\hat{U}_0(t) \qquad \hat{U}_0(0) = \hat{I} \tag{2.5}$$

$$\mathrm{i}\hbar\frac{\partial \hat{U}_I(t)}{\partial t} = \hat{H}_I(t)\hat{U}_I(t) \qquad \hat{U}_I(0) = \hat{I} \tag{2.6}$$

where

$$\hat{H}_I(t) = \hat{U}_0^+(t)\hat{V}(t)\hat{U}_0(t). \tag{2.7}$$

The 'unperturbed' operator $\hat{U}_0(t)$ admits the WN form

$$\hat{U}_0(t) = \mathrm{e}^{\Lambda(t)}\mathrm{e}^{a(t)q^2}\mathrm{e}^{b(t)q\partial/\partial q}\mathrm{e}^{c(t)\partial^2/\partial q^2} \tag{2.8}$$

where the WN characteristic functions $\Lambda(t)$, $a(t)$, $b(t)$, $c(t)$ are given in analytic form in [20]. Replacing (2.8) in (2.7) yields

$$\hat{H}_I(t) = K(t)q - \mathrm{i}N(t)\partial/\partial q \tag{2.9}$$

where

$$K(t) = \mathrm{e}^{-b(t)}[\mu(t) - 2\mathrm{i}\hbar\nu(t)a(t)] \tag{2.10a}$$

$$N(t) = \hbar\nu(t)\mathrm{e}^{b(t)} - 2\mathrm{i}c(t)K(t). \tag{2.10b}$$

Then, by equation (2.6) and, on account of the $h(4)$ structure of $\hat{H}_I(t)$, we get

$$\hat{U}_I(t) = \mathrm{e}^{h(t)q}\mathrm{e}^{f(t)\partial/\partial q}\mathrm{e}^{g(t)} \tag{2.11}$$

where

$$h(t) = -\frac{\mathrm{i}}{\hbar}\int_0^t K(t')\,\mathrm{d}t' \tag{2.12}$$

$$f(t) = -\frac{\mathrm{i}}{\hbar}\int_0^t N(t')\,\mathrm{d}t' \tag{2.13}$$

$$g(t) = \int_0^t h(t')\dot{f}(t')\,\mathrm{d}t' \tag{2.14}$$

(the dot denoting time derivative).

Equations (2.4), (2.8), (2.11)–(2.14) fully solve our problem. Once the evolution operator is known, we can obtain the propagator according to the relation

$$G(q,q',t) = \hat{U}(t)\delta(q-q'). \tag{2.15}$$

We have explicitly

$$\begin{aligned} G(q,q',t) &= \hat{U}(t)\frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp[\mathrm{i}k(q-q')]\,\mathrm{d}k \\ &= \frac{1}{2[\pi c(t)]^{12}} \exp\left[\tilde{\Lambda}(t) + a(t)q^2 + h(t)\mathrm{e}^{b(t)}q + c(t)h^2(t)\right] \\ &\quad \times \exp\left\{-\frac{1}{4c(t)}\left[\mathrm{e}^{b(t)}q - q' + f(t) + 2h(t)c(t)\right]^2\right\} \end{aligned} \tag{2.16}$$

where $\tilde{\Lambda}(t) = \lambda(t) + g(t)$. The exact form (2.16) of the propagator for the generalized quadratic parametric oscillator is derived here apparently for the first time (however, see [21], in which the special case $Y(t) = \nu(t) = 0$ in equation (1.2) has been considered by the same approach).

3. Solution of the inverted, driven CK oscillator

The driven, inverted CK Hamiltonian (1.1) is obviously a special case of (2.1)–(2.3) with

$$Z(t) = \mathrm{e}^{-\gamma t} \qquad Y(t) = 0 \qquad X(t) = -\mathrm{e}^{-\gamma t} \qquad \mu(t) = \lambda\cos\tilde{\Omega}t \qquad \nu(t) = 0. \tag{3.1}$$

Therefore, from [20] and equations (2.12)–(2.14), we obtain the following expression for the WN characteristic functions $a(t)$, $b(t)$, $c(t)$, $\Lambda(t)$, $h(t)$, $f(t)$:

$$a(t) = \frac{\mathrm{i}m\omega}{2\hbar}\frac{\sinh\Omega t}{\cosh(\Omega t+\varphi)}\mathrm{e}^{\gamma t} \tag{3.2}$$

$$b(t) = \frac{\gamma t}{2} - \ln\frac{\cosh(\Omega t+\varphi)}{\cosh\varphi} \tag{3.3}$$

$$c(t) = \frac{\mathrm{i}\hbar}{2m\omega}\frac{\sinh\Omega t}{\cosh(\Omega t+\varphi)} \tag{3.4}$$

$$\Lambda(t) = \tfrac{1}{2}b(t) \tag{3.5}$$

$$h(t) = \frac{\mathrm{i}\lambda}{4\hbar\cosh\varphi}W(t) \tag{3.6}$$

$$f(t) = \frac{1}{4m\omega\cosh\varphi}R(t) \tag{3.7}$$

where

$$\Omega^2 = \omega^2 + \tfrac{1}{4}\gamma^2 \qquad \varphi = \tanh^{-1}\gamma/2\Omega \qquad \cosh\varphi = \Omega/\omega \tag{3.8}$$

and

$$W(t) = \frac{e^{\varphi}\left[1 - e^{(\Omega + i\tilde{\Omega} - \gamma/2)t}\right]}{\Omega + i\tilde{\Omega} - \gamma/2} - \frac{e^{-\varphi}\left[1 - e^{-(\Omega - i\tilde{\Omega} + \gamma/2)t}\right]}{\Omega - i\tilde{\Omega} + \gamma/2} + \frac{e^{\varphi}\left[1 - e^{(\Omega - i\tilde{\Omega} - \gamma/2)t}\right]}{\Omega - i\tilde{\Omega} - \gamma/2} - \frac{e^{-\varphi}\left[1 - e^{-(\Omega + i\tilde{\Omega} + \gamma/2)t}\right]}{\Omega + i\tilde{\Omega} + \gamma/2} \tag{3.9}$$

$$R(t) = \frac{1 - e^{(\Omega + i\tilde{\Omega} - \gamma/2)t}}{\Omega + i\tilde{\Omega} - \gamma/2} + \frac{1 - e^{-(\Omega - i\tilde{\Omega} + \gamma/2)t}}{\Omega - i\tilde{\Omega} - \gamma/2} + \frac{1 - e^{(\Omega - i\tilde{\Omega} - \gamma/2)t}}{\Omega - i\tilde{\Omega} - \gamma/2} + \frac{1 - e^{-(\Omega + i\tilde{\Omega} + \gamma/2)t}}{\Omega + i\tilde{\Omega} + \gamma/2}. \tag{3.10}$$

In order to find the exact wavefunction, according to the formula

$$\psi(q, t) = \int G(q, q', t)\psi(q', 0)\, dq' \tag{3.11}$$

we take the Gaussian wavepacket as the initial state

$$\psi(q', 0) = \frac{1}{(2\pi\sigma^2)^{1/4}} \exp\left[-\frac{(q' - q_0)^2}{4\sigma^2} + ikq'\right] \qquad (k = p_0/\hbar). \tag{3.12}$$

Thus, we get

$$\begin{aligned}\psi(q, t) = {} & \frac{1}{(\sqrt{2\pi}\sigma)^{1/2}} \frac{1}{\sqrt{c(t)/\sigma^2 + 1}} \exp\left[\tilde{\Lambda}(t) + a(t)q^2 + h(t)e^{b(t)}q + c(t)h^2(t)\right] \\ & \times \exp\left\{-\frac{q_0^2}{4\sigma^2} - \frac{1}{4c(t)}\left[e^{b(t)}q + f(t) + 2c(t)h(t)\right]^2\right\} \\ & \times \exp\left\{\left[\frac{q_0}{2\sigma^2} + ik + \frac{1}{2c(t)}\left[e^{b(t)}q + f(t) + 2c(t)h(t)\right]\right]^2 \Big/ \left(\frac{1}{\sigma^2} + \frac{1}{c(t)}\right)\right\}\end{aligned} \tag{3.13}$$

where the WN characteristic functions are given by equations (3.2)–(3.7).

4. Tunnelling process for the inverted driven CK oscillator

We now want to apply the results of the previous section to discuss quantum tunnelling for a driven inverted CK oscillator. As is well known, different definitions of tunnelling times exist in the literature [22]. We do not face this problem here, and rely on the discussion and definitions given in [23].

Therefore, we define as the mean total sojourn (or dwell) time the quantity

$$\tau[(d_1, d_2); -\infty, +\infty; \psi] = \int_{-\infty}^{+\infty} dt\, Q(t) \tag{4.1}$$

where $Q(t)$ is the persistence probability, defined by [24]

$$Q(t) = \int_{d_1}^{d_2} \rho(q, t)\,\mathrm{d}q \tag{4.2}$$

where $\rho(q, t) = |\psi(q, t)|^2$ is the probability density and (d_1, d_2) is an interval containing the barrier region. Let us evaluate the above quantities for the driven, inverted CK system.

After some tedious but straightforward algebra we find the following expression of the probability density

$$\rho(q, t) = |\psi(q, t)|^2 = \frac{1}{\sqrt{2\pi\sigma^2\Gamma^2(t)}} \exp\left[-\frac{(q - q_{\max})^2}{2\sigma^2\Gamma^2(t)} \right] \tag{4.3}$$

where

$$q_{\max} = \mathrm{e}^{-b(t)}[q_0 - 2ikc(t) - f(t) - 2h(t)c(t)] \tag{4.4}$$

$$\Gamma^2(t) = \mathrm{e}^{-2b(t)} \left[1 + \left(\frac{l_0}{\sigma}\right)^4 \frac{\sinh^2 \Omega t}{\cosh^2(\Omega t + \varphi)} \right] \tag{4.5}$$

and $l_0 = (\hbar/2m\omega)^{1/2}$ is a scattering length characteristic of the system.

Of course, for $\gamma \to 0$ and $\lambda \to 0$ one recovers exactly the results of [25] for the inverted harmonic oscillator.

As to the persistence probability, we get

$$Q(t) = \frac{1}{2}\left[\mathrm{Erf}\left(\frac{d_2 - q_{\max}}{\sqrt{2}\sigma\Gamma(t)} \right) - \mathrm{Erf}\left(\frac{d_1 - q_{\max}}{\sqrt{2}\sigma\Gamma(t)} \right) \right] \tag{4.6}$$

where Erf[·] is the error function.

Let us now calculate the sojourn time. Replacing equation (4.6) in equation (4.1) yields

$$\tau[(d_1, d_2); -\infty, +\infty; \psi] = \frac{1}{2} \int_{-\infty}^{+\infty} \left[\mathrm{Erf}\left(\frac{d_2 - q_{\max}}{\sqrt{2}\sigma\Gamma(t)} \right) - \mathrm{Erf}\left(\frac{d_1 - q_{\max}}{\sqrt{2}\sigma\Gamma(t)} \right) \right] \mathrm{d}t. \tag{4.7}$$

The integral on the right-hand side of equation (4.7) cannot be evaluated exactly. However, if the parameter λ of the driving term in Hamiltonian (1.1) is small, we can get an approximate expression for the dwell time. Indeed, expanding the error functions in powers of λ yields, to first order, the following expression for the integrand in brackets in equation (4.7):

$$\begin{aligned} &\left[\mathrm{Erf}\left(\frac{d_2 - q_{\max}(\lambda = 0)}{\sqrt{2}\sigma\Gamma(t)} \right) - \mathrm{Erf}\left(\frac{d_1 - q_{\max}(\lambda = 0)}{\sqrt{2}\sigma\Gamma(t)} \right) \right] \\ &\qquad + \frac{\lambda\sqrt{2/\pi}}{\sigma\Gamma(t)} \frac{\mathrm{e}^{-b(t)}}{4m\omega\cosh\varphi} \left[R(t) - \frac{\sinh\Omega t}{\cosh(\Omega t + \varphi)} W(t) \right] \\ &\qquad \times \left\{ \exp\left[-\left(\frac{d_2 - q_{\max}(\lambda = 0)}{\sqrt{2}\sigma\Gamma(t)} \right)^2 \right] - \exp\left[-\left(\frac{d_1 - q_{\max}(\lambda = 0)}{\sqrt{2}\sigma\Gamma(t)} \right)^2 \right] \right\}. \end{aligned} \tag{4.8}$$

Hence the sojourn time takes the form

$$\tau[(d_1,d_2);-\infty,+\infty;\psi] = \frac{1}{2}\int_{-\infty}^{+\infty}\left\{\mathrm{Erf}\left[\frac{d_2-q_{\max}(\lambda=0)}{\sqrt{2}\sigma\Gamma(t)}\right]-\mathrm{Erf}\left[\frac{d_1-q_{\max}(\lambda=0)}{\sqrt{2}\sigma\Gamma(t)}\right]\right\}\mathrm{d}t$$
$$+\frac{\lambda}{8m\omega\sigma}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{+\infty}\frac{I(t)}{[1+(l_0/\sigma)^4(\sinh^2\Omega t/\cosh^2(\Omega t+\varphi))]^{1/2}\cosh(\omega t+\varphi)}$$
$$\times\left\{\exp\left[-\left(\frac{d_2-q_{\max}(\lambda=0)}{\sqrt{2}\sigma\Gamma(t)}\right)^2\right]-\exp\left[-\left(\frac{d_1-q_{\max}(\lambda=0)}{\sqrt{2}\sigma\Gamma(t)}\right)^2\right]\right\}\mathrm{d}t \tag{4.9}$$

where

$$I(t) = \left\{(\Omega+\tfrac{1}{2}\gamma)\left[2\mathrm{e}^{\Omega t}-\left(\mathrm{e}^{(\mathrm{i}\tilde{\Omega}-\gamma/2)t}+\mathrm{e}^{-(\mathrm{i}\tilde{\Omega}+\gamma/2)t}\right)\right]-\mathrm{i}\tilde{\Omega}\left(\mathrm{e}^{(\mathrm{i}\tilde{\Omega}-\gamma/2)t}-\mathrm{e}^{-(\mathrm{i}\tilde{\Omega}+\gamma/2)t}\right)\right\}$$
$$\times\frac{1}{(\Omega+\gamma/2)^2+\tilde{\Omega}^2}$$
$$+\left\{(\Omega-\tfrac{1}{2}\gamma)\left[2\mathrm{e}^{-\Omega t}-\left(\mathrm{e}^{(\mathrm{i}\tilde{\Omega}-\gamma/2)t}+\mathrm{e}^{-(\mathrm{i}\tilde{\Omega}+\gamma/2)t}\right)\right]\right.$$
$$\left.+\mathrm{i}\tilde{\Omega}\left(\mathrm{e}^{(\mathrm{i}\tilde{\Omega}-\gamma/2)t}-\mathrm{e}^{-(\mathrm{i}\tilde{\Omega}+\gamma/2)t}\right)\right\}\frac{1}{(\Omega-\gamma/2)^2+\tilde{\Omega}^2}. \tag{4.10}$$

In the case of an extended wavepacket (i.e. with a width much greater than the characteristic length, $\sigma \gg l_0$), it is possible to show (see the appendix) that the sojourn time (4.9) reads explicitly

$$\tau[(d_1,d_2);-\infty,+\infty;\psi] = \frac{\pi}{2\omega}\frac{d_2-d_1}{\sqrt{2}\sigma}f(\gamma,\omega)+\lambda F(\gamma,\omega,\tilde{\Omega}) \tag{4.11}$$

where (see equation (A.4))

$$f(\gamma,\omega) = \exp\left(-\frac{\gamma}{2\Omega}\tanh^{-1}\frac{\gamma}{2\Omega}\right)\cos^{-1}\left(\frac{\gamma\pi}{4\Omega}\right) \tag{4.12}$$

and, for a wavepacket initially centred at the origin ($q_0 = 0$), the function $F(\gamma,\Omega,\tilde{\Omega})$ is given by equation (A.8). We therefore see that the total sojourn time includes two terms. The first one corresponds to the inverted CK Hamiltonian without the perturbation, and is an increasing function of the dissipation parameter γ [18]. In fact, the function $f(\gamma,\omega)$ takes its maximum value $f(\gamma,\omega) = 1$ for $\gamma = 0$. The decrease in the tunnelling process due to the dissipation has also been predicted in a different way by other authors (see, e.g., [26]). The second term of the sojourn time is due to the perturbation and also depends on the frequency $\tilde{\Omega}$ of the external force.

Finally, let us write down the expression for the sojourn time for the special case $\gamma = 0$, i.e. for a driven inverted oscillator with Hamiltonian [24]

$$\hat{H} = \frac{\hat{p}^2}{2m}-\frac{m}{2}\omega^2\hat{q}^2+\lambda\hat{q}\cos\tilde{\Omega}t. \tag{4.13}$$

In this case, we have simply $f(0,\omega) = 1$. The function $F(0,\omega,\tilde{\omega})$ can be obtained by putting $\gamma = 0$ in equation (A.8) or by a direct computation of the integral in equation (4.9). We get

$$\tau = \frac{\pi}{2\omega}\frac{(d_2-d_1)}{\sqrt{2}\sigma}+\frac{1}{\sqrt{2\pi}}\frac{d_1^2-d_2^2}{2\sigma^3}\frac{\lambda}{m\omega(\omega^2+\tilde{\Omega}^2)} \tag{4.14}$$

5. Conclusion

In this paper, we have applied the evolution operator method (and the WN theorem) to obtain the exact propagator of the generalized quadratic parametric oscillator, by exploiting the $SU(1,1) \oplus h(4)$ algebraic structure of its Hamiltonian. The results have been applied to the driven, inverted CK oscillator, taking a Gaussian wavepacket as the initial state. The probability density and persistence probability have been evaluated. The exact expression of the sojourn time is derived. For a small perturbation, the sojourn time to the first order in the perturbative parameter is the sum of two terms, whose expression is found explicitly for the case of an extended wavepacket. The first one (depending on the dissipation parameter and the oscillator frequency) is the sojourn time of the inverted CK oscillator, whereas the second one is due to the presence of the driving force. The special case of the absence of dissipation has also been considered.

Acknowledgments

Two of us (SB and AJ) are grateful to the Department of Physics of the University 'La Sapienza', Rome, for kind hospitality and to the INFN, Sezione di Roma, for technical support. Moreover, the authors thank one of the referees for having drawn their attention to the paper by Lo [21].

Appendix. Calculation of the sojourn time

Let us consider as the initial wavefunction a wavepacket whose width is greater than the characteristic length associated with the driven CK Hamiltonian (1.1), i.e. $\sigma \gg l_0$. In the same hypothesis, we can also assume that

$$|d_{1,2} - q_{\max}(\lambda = 0)| \ll \sigma \qquad d_{1,2} \ll \sigma. \tag{A.1}$$

Then, the error functions in the first term of equation (3.22) can be approximated by

$$\mathrm{Erf}\left[\frac{d_{1,2} - q_{\max}(\lambda = 0)}{\sqrt{2}\sigma\Gamma(t)}\right] \approx \frac{d_{1,2} - q_{\max}(\lambda = 0)}{\sqrt{2}\sigma\Gamma(t)}. \tag{A.2}$$

Moreover, for $\sigma \gg l_0$, from equations (3.15), (3.16) and (3.22) and the expressions (3.2)–(3.6) of the WN functions, it is not difficult to obtain the following approximate expression of the sojourn time τ (valid for an extended wavepacket):

$$\begin{aligned}\tau[(d_1, d_2); -\infty, +\infty; \psi] = {} & \frac{1}{2}\frac{(d_2 - d_1)}{\sqrt{2}\sigma}\cosh\varphi \int_{-\infty}^{+\infty} \frac{\mathrm{e}^{\gamma t/2}}{\cosh(\Omega t + \varphi)}\,\mathrm{d}t \\ & + \lambda\frac{(2/\pi)^{1/2}}{8m\omega\sigma}\int_{-\infty}^{+\infty}\frac{I(t)}{\cosh(\Omega t + \varphi)}\left\{\exp\left[-\left(\frac{d_2\mathrm{e}^{b(t)} - q_0 + 2\mathrm{i}kc(t)}{\sqrt{2}\sigma}\right)^2\right]\right. \\ & \left. - \exp\left[-\left(\frac{d_1\mathrm{e}^{b(t)} - q_0 + 2\mathrm{i}kc(t)}{\sqrt{2}\sigma}\right)^2\right]\right\}\mathrm{d}t. \end{aligned} \tag{A.3}$$

The first integration is straightforward and yields the following expression of the first term

$$\frac{\pi}{2\omega}\frac{(d_2-d_1)}{\sqrt{2}\sigma}\exp\left(-\frac{\gamma}{2\Omega}\tanh^{-1}\frac{\gamma}{2\Omega}\right)\cos^{-1}\left(\frac{\gamma\pi}{4\Omega}\right). \tag{A.4}$$

For the evaluation of the second integral in the above equation we can assume, without loss of generality, that $q_0 = 0$ (i.e. an initial wavepacket centred at the origin) and that k is small. Thus, by still taking into account that $\sigma \gg l_0$, the quantity in curly brackets becomes simply

$$\{\ldots\} \approx \frac{d_1^2 - d_2^2}{2\sigma^2}\mathrm{e}^{2b(t)}. \tag{A.5}$$

Let us denote by $F(\gamma, \omega, \tilde{\Omega}, t)$ the function multiplying the perturbation parameter λ. On account of equation (A.5) and by the above assumptions, we can write

$$\begin{aligned}
F(\gamma,\omega,\tilde{\Omega},t) = {} & \frac{(2/\pi)^{1/2}}{8m\omega\Omega\sigma}\frac{d_1^2-d_2^2}{2\sigma^2}\cosh^2\varphi\left\{\frac{1}{(\Omega+\gamma/2)^2+\tilde{\Omega}^2}\right. \\
& \times\left\{2(\Omega+\gamma/2)\left[\mathrm{e}^{-(1+\gamma/\Omega)\varphi}\int_0^\infty\frac{\cosh(1+\gamma/\Omega)u}{\cosh^3 u}\,\mathrm{d}u\right.\right. \\
& + 2(\Omega+\gamma/2)\mathrm{e}^{-\gamma\varphi/2\Omega}\left[\cos\left(\frac{\tilde{\Omega}}{\Omega}\varphi\right)\int_0^\infty\frac{\cos(\tilde{\Omega}u/\Omega)\cosh(\gamma u/2\Omega)}{\cosh^3 u}\,\mathrm{d}u\right. \\
& \left. + \sin\left(\frac{\tilde{\Omega}}{\Omega}\varphi\right)\int_0^\infty\frac{\sin(\tilde{\Omega}u/\Omega)\sinh(\gamma u/2\Omega)}{\cosh^3 u}\,\mathrm{d}u\right] \\
& + 2\tilde{\Omega}\mathrm{e}^{-\gamma\varphi/2\Omega}\left[\cos\left(\frac{\tilde{\Omega}}{\Omega}\varphi\right)\int_0^\infty\frac{\sinh(\gamma u/2\Omega)\sin(\tilde{\Omega}u/\Omega)}{\cosh^3 u}\,\mathrm{d}u\right. \\
& \left.\left.\left. - \sin\left(\frac{\tilde{\Omega}}{\Omega}\varphi\right)\int_0^\infty\frac{\cosh(\gamma u/2\Omega)\cos(\tilde{\Omega}u/\Omega)}{\cosh^3 u}\,\mathrm{d}u\right]\right]\right\} \\
& + \frac{1}{(\Omega-\gamma/2)^2+\tilde{\Omega}^2}\left\{2(\Omega-\gamma/2)\mathrm{e}^{(1-\gamma/\Omega)\varphi}\int_0^\infty\frac{\cosh(1-\gamma/\Omega)u}{\cosh^3 u}\,\mathrm{d}u\right. \\
& - 2(\Omega-\gamma/2)\mathrm{e}^{\gamma\varphi/2\Omega}\left[\cos\left(\frac{\tilde{\Omega}}{\Omega}\varphi\right)\int_0^\infty\frac{\cos(\tilde{\Omega}u/\Omega)\cosh(\gamma u/2\Omega)}{\cosh^3 u}\,\mathrm{d}u\right. \\
& \left. + \sin\left(\frac{\tilde{\Omega}}{\Omega}\varphi\right)\int_0^\infty\frac{\sin(\tilde{\Omega}u/\Omega)\sinh(\gamma u/2\Omega)}{\cosh^3 u}\,\mathrm{d}u\right] \\
& + 2\tilde{\Omega}\mathrm{e}^{-\gamma\varphi/2\Omega}\left[\sin\left(\frac{\tilde{\Omega}}{\Omega}\varphi\right)\int_0^\infty\frac{\cos(\tilde{\Omega}u/\Omega)\cosh(\gamma u/2\Omega)}{\cosh^3 u}\,\mathrm{d}u\right. \\
& \left.\left.\left. - \cos\left(\frac{\tilde{\Omega}}{\Omega}\varphi\right)\int_0^\infty\frac{\sin(\tilde{\Omega}u/\Omega)\sinh(\gamma u/2\Omega)}{\cosh^3 u}\,\mathrm{d}u\right]\right\}\right\}.
\end{aligned} \tag{A.6}$$

Exploiting the well known formula

$$\int_0^\infty\frac{\cosh(2\beta u)}{\cosh^{2\nu}(\alpha u)}\,\mathrm{d}u = 4^{\nu-1}B\left(\nu+\frac{\beta}{\alpha},\nu-\frac{\beta}{\alpha}\right) \qquad \mathrm{Re}(\nu\pm\beta)>0 \qquad \alpha>0 \tag{A.7}$$

where $B(x, y)$ is the beta function (Euler's integral of the first kind), equation (A.6) takes the form

$$F(\gamma,\omega,\tilde{\Omega},t)=\frac{(2/\pi)^{1/2}}{8m\omega\Omega\sigma}\frac{d_1^2-d_2^2}{2\sigma^2}\cosh^2\varphi\bigg\{\frac{1}{(\Omega+\gamma/2)^2+\tilde{\Omega}}$$
$$\times\bigg\{4(\Omega+\gamma/2)\mathrm{e}^{(-1+\gamma/\Omega)\varphi}B(2+\gamma/2\Omega,1-\gamma/2\Omega)+2(\Omega+\gamma/2)\mathrm{e}^{-\gamma\varphi/2\Omega}$$
$$\times\bigg\{\cos\left(\frac{\tilde{\Omega}}{\Omega}\varphi\right)[B(\tfrac{3}{2}+\tfrac{1}{2}\mu,\tfrac{3}{2}-\tfrac{1}{2}\mu)+B(\tfrac{3}{2}+\tfrac{1}{2}\mu*,\tfrac{3}{2}-\tfrac{1}{2}\mu*)]$$
$$-\mathrm{i}\sin\left(\frac{\tilde{\Omega}}{\Omega}\varphi\right)[B(\tfrac{3}{2}+\tfrac{1}{2}\mu,\tfrac{3}{2}-\tfrac{1}{2}\mu)-B(\tfrac{3}{2}+\tfrac{1}{2}\mu*,\tfrac{3}{2}-\tfrac{1}{2}\mu*)]\bigg\}$$
$$+2\tilde{\Omega}\mathrm{e}^{-\gamma\varphi/2\Omega}\bigg\{-\mathrm{i}\cos\left(\frac{\tilde{\Omega}}{\Omega}\varphi\right)[B(\tfrac{3}{2}+\tfrac{1}{2}\mu,\tfrac{3}{2}-\tfrac{1}{2}\mu)$$
$$-B(\tfrac{3}{2}+\tfrac{1}{2}\mu*,\tfrac{3}{2}-\tfrac{1}{2}\mu*)]-\sin\left(\frac{\tilde{\Omega}}{\Omega}\varphi\right)[B(\tfrac{3}{2}+\tfrac{1}{2}\mu,\tfrac{3}{2}-\tfrac{1}{2}\mu)$$
$$+B(\tfrac{3}{2}+\tfrac{1}{2}\mu*,\tfrac{3}{2}-\tfrac{1}{2}\mu*)]\bigg\}\bigg\}$$
$$+\frac{1}{(\Omega-\gamma/2)+\tilde{\Omega}^2}\bigg\{4(\Omega-\gamma/2)\mathrm{e}^{(1-\gamma/\Omega)\varphi}B(2-\gamma/2\Omega,1+\gamma/2\Omega)$$
$$-2(\Omega-\gamma/2)\mathrm{e}^{-\gamma\varphi/2\Omega}\bigg\{\cos\left(\frac{\tilde{\Omega}}{\Omega}\varphi\right)[B(\tfrac{3}{2}+\tfrac{1}{2}\mu,\tfrac{3}{2}-\tfrac{1}{2}\mu)$$
$$+B(\tfrac{3}{2}+\tfrac{1}{2}\mu*,\tfrac{3}{2}-\tfrac{1}{2}\mu*)]-\mathrm{i}\sin\left(\frac{\tilde{\Omega}}{\Omega}\varphi\right)[B(\tfrac{3}{2}+\tfrac{1}{2}\mu,\tfrac{3}{2}-\tfrac{1}{2}\mu)$$
$$-B(\tfrac{3}{2}+\tfrac{1}{2}\mu*,\tfrac{3}{2}-\tfrac{1}{2}\mu*)]\bigg\}+2\tilde{\Omega}\mathrm{e}^{-\gamma\varphi/\Omega}\bigg\{\sin\left(\frac{\tilde{\Omega}}{\Omega}\varphi\right)[B(\tfrac{3}{2}+\tfrac{1}{2}\mu,\tfrac{3}{2}-\tfrac{1}{2}\mu)$$
$$+B(\tfrac{3}{2}+\tfrac{1}{2}\mu*,\tfrac{3}{2}-\tfrac{1}{2}\mu*)]+\mathrm{i}\cos\left(\frac{\tilde{\Omega}}{\Omega}\varphi\right)$$
$$\times[B(\tfrac{3}{2}+\tfrac{1}{2}\mu,\tfrac{3}{2}-\tfrac{1}{2}\mu)-B(\tfrac{3}{2}+\tfrac{1}{2}\mu*,\tfrac{3}{2}-\tfrac{1}{2}\mu*)]\bigg\}\bigg\}\bigg\} \tag{A.8}$$

where $\mu=\gamma/2\Omega+\mathrm{i}\tilde{\Omega}/\Omega$, and the star denotes the usual conjugation of complex numbers.

References

[1] See, e.g., Exners P 1984 *Open Quantum Systems and Feynman Integrals* (Dordrecht: Reidel)
Khandekar D C and Lawande S V 1986 *Phys. Rep.* **137** 117 and references therein

[2] Lewis H R and Riesenfeld W B 1969 *J. Math. Phys.* **10** 1458

[3] See, e.g., Dodonov V and Man'ko V 1989 *Invariants and the Evolution of Nonstationary Quantum Systems* ed M Markov (Commack, NY: Nova Science) and references therein

[4] Camiz P, Gerardi A, Marchioro C, Presutti E and Scacciatelli E 1971 *J. Math. Phys.* **12** 2040

[5] Oh H G, Lee H R, George T F and Um C I 1989 *Phys. Rev.* A **39** 5515 and references therein

[6] Dattoli G, Solimeno S and Torre A 1986 *Phys. Rev.* A **34** 2646
[7] See e.g. Dattoli G, Gallardo T and Torre A 1988 *Riv. Nuovo Cimento* **11** 1
[8] Cheng C M and Fung P C W 1988 *J. Phys. A: Math. Gen.* **21** 4115
Lo C F 1990 *Nuovo Cimento* B **105** 497
[9] Wei J and Norman E 1963 *J. Math. Phys.* 4 575
[10] Dattoli G, Orsitto F and Torre A 1986 *Phys. Rev.* A **34** 2566
Dattoli G, Torre A and Richetta M 1988 *Phys. Rev.* A **37** 2007
[11] Lo C F 1991 *Nuovo Cimento* D **13** 1279
[12] Ben-Aryeh Y and Mann A 1985 *Phys. Rev. Lett.* **54** 1020
[13] Caldirola P 1941 *Nuovo Cimento* **13** 393
Kanai E 1948 *Prog. Theor. Phys.* **3** 440
[14] Baskoutas S and Jannussis A 1992 *Nuovo Cimento* B **107** 255
[15] Murray S III, Scully M D and Lamb Ir W E 1974 *Laser Physics* (New York: Addison-Wesley) p 61
[16] Salbi N A, Kouri D I, Baer M and Pollok E 1985 *J. Chem. Phys.* **82** 4500
[17] Jannussis A and Skaltsas D 1990 *Hadr. J.* **13** 89
[18] Baskoutas S and Jannussis A 1992 *J. Phys. A: Math. Gen.* **25** L1299
[19] Wolf K B 1981 *J. Appl. Math.* **40** 419
[20] Baskoutas S, Jannussis A and Mignani R 1992 *Phys. Lett.* A **164** 17
[21] Lo C F 1993 *Phys. Rev.* A **47** 115
[22] For a recent review, see Olkhovsky V S and Recami E 1992 *Phys. Rep.* **214** 339
[23] Jaworsky W and Wardlaw D 1991 *Phys. Rev.* A **43** 5137 and references therein
See also Hauge E H and Stovneng I A 1989 *Rev. Mod. Phys.* **61** 917
[24] Barton G 1986 *Ann. Phys., NY* **166** 322
[25] Papadopoulus G I 1990 *J. Phys. A: Math. Gen.* **23** 935
[26] Büttiker M and Landauer R 1982 *Phys. Rev. Lett.* **49** 1739; 1988 *J. Phys. C: Solid State Phys.* **21** 6207